

THE TOPOLOGY OF ULTRAFILTERS AS SUBSPACES OF 2^ω

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ABSTRACT. Using the property of being completely Baire, countable dense homogeneity and the perfect set property we will be able, under Martin's Axiom for countable posets, to distinguish non-principal ultrafilters on ω up to homeomorphism. Here, we identify ultrafilters with subspaces of 2^ω in the obvious way. Using the same methods, still under Martin's Axiom for countable posets, we will construct a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that \mathcal{U}^ω is countable dense homogeneous. This consistently answers a question of Hrušák and Zamora Avilés. Finally, we will give some partial results about the relation of such topological properties with the combinatorial property of being a P-point.

By identifying a subset of ω with an element of the Cantor set 2^ω in the obvious way (which we will freely do throughout the paper), it is possible to study the topological properties of any $\mathcal{X} \subseteq \mathcal{P}(\omega)$. We will focus on the case $\mathcal{X} = \mathcal{U}$, where \mathcal{U} is an ultrafilter on ω . The case $\mathcal{X} = \mathcal{F}$, where \mathcal{F} is simply a filter on ω , has been studied extensively (see Chapter 4 in [3]). From now on, all filters and ideals are implicitly assumed to be on ω .

First, we will observe that there are many (actually, as many as possible) non-homeomorphic ultrafilters. However, the proof is based on a cardinality argument, hence it is not 'honest' in the sense of Van Douwen: it would be desirable to find 'quotable' topological properties that distinguish ultrafilters up to homeomorphism. This is consistently achieved in Section 3 using the property of being completely Baire (see Corollary 9 and Theorem 11), in Section 4 using countable dense homogeneity (see Theorem 15 and Theorem 21) and in Section 6 using the perfect set property (see Theorem 28 and Corollary 31).

In Section 5, we will adapt the proof of Theorem 21 to obtain the countable dense homogeneity of the ω -power, consistently answering a question of Hrušák and Zamora Avilés from [10] (see Corollary 26).

In Section 7, using a modest large cardinal assumption, we will obtain a strong generalization of the main result of Section 6 (see Theorem 35).

Finally, in Section 8, we will investigate the relationship between the property of being a P-point and the above topological properties; many questions on this front remain open.

Proposition 1. *Let $\mathcal{U}, \mathcal{V} \subseteq 2^\omega$ be non-principal ultrafilters. Define $\mathcal{U} \cong \mathcal{V}$ if the topological spaces \mathcal{U} and \mathcal{V} are homeomorphic. Then the equivalence classes of \cong have size \mathfrak{c} .*

Proof. To show that each equivalence class has size at least \mathfrak{c} , simply use homeomorphisms of 2^ω induced by permutations of ω and an almost disjoint family of subsets of ω of size \mathfrak{c} (see, for example, Lemma 9.21 in [12]).

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By Lavrentiev's lemma (see Theorem 3.9 in [13]), if $g : \mathcal{U} \longrightarrow \mathcal{V}$ is a homeomorphism, then there exists a homeomorphism $f : G \longrightarrow H$ that extends g , where G and H are G_δ subsets of 2^ω . Since there are only \mathfrak{c} such homeomorphisms, it follows that an equivalence class of \cong has size at most \mathfrak{c} . \square

Corollary 2. *There are $2^\mathfrak{c}$ pairwise non-homeomorphic non-principal ultrafilters.*

1. NOTATION AND TERMINOLOGY

Our main reference for descriptive set theory is [13]. For other set-theoretic notions, see [3] or [12]. For notions that are related to large cardinals, see [14]. For all undefined topological notions, see [6].

By *space* we mean separable metrizable topological space, with a unique exception in Section 6. For every $s \in {}^{<\omega}2$, we will denote by $[s]$ the basic clopen set $\{x \in 2^\omega : s \subseteq x\}$. Given a tree $T \subseteq {}^{<\omega}2$, we will denote by $[T]$ the set of branches of T , that is $[T] = \{x \in 2^\omega : x \upharpoonright n \in T \text{ for all } n \in \omega\}$.

Given a function f and $A \subseteq \text{dom}(f)$, we will denote by $f[A]$ the image of A under f , that is $f[A] = \{f(x) : x \in A\}$.

A space X is *homogeneous* if whenever $x, y \in X$ there exists a homeomorphism $f : X \longrightarrow X$ such that $f(x) = y$.

Define the homeomorphism $c : 2^\omega \longrightarrow 2^\omega$ by setting $c(x)(n) = 1 - x(n)$ for every $x \in 2^\omega$ and $n \in \omega$. Using c , one sees that every ultrafilter $\mathcal{U} \subseteq 2^\omega$ is homeomorphic to its dual maximal ideal $\mathcal{J} = 2^\omega \setminus \mathcal{U} = c[\mathcal{U}]$.

A *perfect set* in a space X is a non-empty closed subset P of X with no isolated points. Recall that P is a perfect set in 2^ω if and only if it is homeomorphic to 2^ω . A *Bernstein set* is a subset B of $X = 2^\omega$ such that B and $X \setminus B$ both intersect every perfect set in X . Given such a set B , since 2^ω is homeomorphic to $2^\omega \times 2^\omega$, one actually has $|P \cap B| = \mathfrak{c}$ and $|P \cap (X \setminus B)| = \mathfrak{c}$ for every perfect set P in X .

For every $x \subseteq \omega$, define $x^0 = \omega \setminus x$ and $x^1 = x$. Given a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$, a *word* in \mathcal{A} is an intersection of the form

$$\bigcap_{x \in \tau} x^{w(x)}$$

for some $\tau \in [\mathcal{A}]^{<\omega}$ and $w : \tau \longrightarrow 2$. Recall that \mathcal{A} is an *independent family* if every word in \mathcal{A} is infinite.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ has the *finite intersection property* if $\bigcap \sigma$ is infinite for all $\sigma \in [\mathcal{F}]^{<\omega}$. Given such a family, we will denote by $\langle \mathcal{F} \rangle$ the filter generated by \mathcal{F} . Let Cof be the collection of all cofinite subsets of ω . Recall that an ultrafilter \mathcal{U} is non-principal if and only if $\text{Cof} \subseteq \mathcal{U}$. In particular, every non-principal ultrafilter is dense in 2^ω . For any fixed $x \in 2^\omega$, define $x \upharpoonright = \{y \in 2^\omega : x \subseteq y\}$.

Whenever $x, y \in \mathcal{P}(\omega)$, define $x \subseteq^* y$ if $x \setminus y$ is finite. Given $\mathcal{C} \subseteq \mathcal{P}(\omega)$, a *pseudointersection* of \mathcal{C} is a subset x of ω such that $x \subseteq^* y$ for all $y \in \mathcal{C}$. Given a cardinal κ , a non-principal ultrafilter \mathcal{U} is a P_κ -point if every $\mathcal{C} \in [\mathcal{U}]^{<\kappa}$ has a pseudointersection in \mathcal{U} . A P -point is simply a P_{ω_1} -point.

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ has the *finite union property* if $\bigcup \sigma$ is coinfinite for all $\sigma \in [\mathcal{I}]^{<\omega}$. Given such a family, we will denote by $\langle \mathcal{I} \rangle$ the ideal generated by \mathcal{I} . Let Fin be the collection of all finite subsets of ω . For any fixed $x \in 2^\omega$, define $x \downharpoonright = \{y \in 2^\omega : y \subseteq x\}$.

Given $\mathcal{C} \subseteq \mathcal{P}(\omega)$, a *pseudounion* of \mathcal{C} is a subset x of ω such that $y \subseteq^* x$ for all $y \in \mathcal{C}$. A maximal ideal \mathcal{J} is a P -ideal if $c[\mathcal{J}]$ is a P -point.

2. BASIC PROPERTIES

In this section, we will notice that some topological properties are shared by all non-principal ultrafilters. It is easy to realize that every principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ is homeomorphic to 2^ω .

Since any maximal ideal \mathcal{J} (actually, any ideal) is a topological subgroup of 2^ω under the operation of symmetric difference (or equivalently, sum modulo 2), every ultrafilter $\mathcal{U} = c[\mathcal{J}]$ is also a topological group. In particular, every ultrafilter \mathcal{U} is a homogeneous topological space.

The following proposition is Lemma 3.1 in [8].

Proposition 3 (Fitzpatrick, Zhou). *Let X be a homogeneous topological space. Then X is a Baire space if and only if X is not meager in itself.*

Proof. One implication is trivial. Now assume that X is not a Baire space. Since X is homogeneous, it follows easily that

$$\mathcal{B} = \{U : U \text{ is a non-empty meager open set in } X\}$$

is a base for X . So $X = \bigcup \mathcal{B}$ is the union of a collection of meager open sets. Hence X is meager by Banach's category theorem (see Theorem 16.1 in [20]).

For the convenience of the reader, we sketch the proof in our particular case. Fix a maximal $\mathcal{C} \subseteq \mathcal{B}$ consisting of pairwise disjoint sets. Observe that $X \setminus \bigcup \mathcal{C}$ is closed nowhere dense. For every $U \in \mathcal{C}$, fix nowhere dense sets $N_n(U)$ such that $U = \bigcup_{n \in \omega} N_n(U)$. It is easy to check that $\bigcup_{U \in \mathcal{C}} N_n(U)$ is nowhere dense in X for every $n \in \omega$. \square

Given any ultrafilter $\mathcal{U} \subseteq 2^\omega$, notice that c is a homeomorphism of 2^ω such that 2^ω is the disjoint union of \mathcal{U} and $c[\mathcal{U}]$. In particular, \mathcal{U} must be non-meager and non-comeager in 2^ω by Baire's category theorem. Actually, it follows easily from the 0-1 Law that no non-principal ultrafilter \mathcal{U} can have the property of Baire (see Theorem 8.47 in [13]). In particular, no non-principal ultrafilter \mathcal{U} can be analytic (see Theorem 21.6 in [13]) or co-analytic.

Corollary 4. *Let $\mathcal{U} \subseteq 2^\omega$ be an ultrafilter. Then \mathcal{U} is a Baire space.*

Proof. If \mathcal{U} were meager in itself, then it would be meager in 2^ω , which is a contradiction. \square

On the other hand, by Theorem 8.17 in [13], no non-principal ultrafilter can be a Choquet space (see Section 8.C in [13]).

3. COMPLETELY BAIRE ULTRAFILTERS

Definition 5. A space X is *completely Baire* if every closed subspace of X is a Baire space.

For example, every Polish space is completely Baire. For co-analytic spaces, the converse is also true (see Corollary 21.21 in [13]).

In the proof of Theorem 11, we will need the following characterization (see Corollary 1.9.13 in [16]). Observe that one implication is trivial.

Lemma 6 (Hurewicz). *A space is completely Baire if and only if it does not contain any closed homeomorphic copy of \mathbb{Q} .*

The following (well-known) lemma is the first step in constructing an ultrafilter that is not completely Baire.

Lemma 7. *There exists a perfect subset P of 2^ω such that P is an independent family.*

Proof. We will give three proofs. The first proof simply shows that the classical construction of an independent family of size \mathfrak{c} (see for example Lemma 7.7 in [12]) actually gives a perfect independent family. Define

$$I = \{(\ell, F) : \ell \in \omega, F \subseteq {}^\ell 2\}.$$

Since I is a countably infinite set, we can identify 2^I and 2^ω . The desired independent family will be a collection of subsets of I . Consider the function $f : 2^\omega \rightarrow 2^I$ defined by

$$f(x) = \{(\ell, F) : x \restriction \ell \in F\}.$$

It is easy to check that f is a continuous injection, hence a homeomorphic embedding by compactness. It follows that $P = \text{ran}(f)$ is a perfect set. To check that P is an independent family, fix $\tau \in [P]^{<\omega}$ and $w : \tau \rightarrow 2$. Suppose that $\tau = f[\sigma]$, where $\sigma = \{x_1, \dots, x_k\}$ and x_1, \dots, x_k are distinct. Choose ℓ large enough so that $x_1 \restriction \ell, \dots, x_k \restriction \ell$ are distinct. It follows that

$$(\ell', \{x \restriction \ell' : x \in \sigma \text{ and } w(f(x)) = 1\}) \in \bigcap_{y \in \tau} y^{w(y)}$$

for every $\ell' \geq \ell$, which concludes the proof.

The second proof is also combinatorial. We will inductively construct $k_n \in \omega$ and a finite tree $T_n \subseteq {}^{<\omega}2$ for every $n \in \omega$ so that the following conditions are satisfied.

- (1) $k_m < k_n$ whenever $m < n < \omega$.
- (2) $T_m \subseteq T_n$ whenever $m \leq n < \omega$.
- (3) All maximal elements of T_n have length k_n . We will use the notation $M_n = \{t \in T_n : \text{dom}(t) = k_n\}$.
- (4) For every $t \in T_n$ there exist two distinct elements of T_{n+1} whose restriction to k_n is t .
- (5) Given any $v : M_n \rightarrow 2$, there exists $i \in k_{n+1} \setminus k_n$ such that $t(i) = v(t \restriction k_n)$ for every $t \in M_{n+1}$.

In the end, set $T = \bigcup_{n < \omega} T_n$ and $P = [T]$. Condition (4) guarantees that P is perfect. Next, we will verify that condition (5) guarantees that P is an independent family. Fix $\tau \in [P]^{<\omega}$ and $w : \tau \rightarrow 2$. For all sufficiently large $n \in \omega$, some $v \in {}^{M_n}2$ satisfies $v(x \restriction k_n) = w(x)$ for all $x \in \tau$. By condition (5), there exists $i \in k_{n+1} \setminus k_n$ such that

$$x(i) = (x \restriction k_{n+1})(i) = v(x \restriction k_n) = w(x)$$

for all $x \in \tau$.

Start with $k_0 = 0$ and $T_0 = \{\emptyset\}$. Given k_n and T_n , define $k_{n+1} = k_n + 2^{|M_n|} + 1$. Fix an enumeration $\{v_j : j \in 2^{|M_n|}\}$ of all functions $v : M_n \rightarrow 2$. Let T_{n+1} consist of all initial segments of functions $t : k_{n+1} \rightarrow 2$ such that $t \restriction k_n \in M_n$ and $t(k_n + j) = v_j(t \restriction k_n)$ for all $j < 2^{|M_n|}$. Then, condition (5) is clearly satisfied. Since there is no restriction on $t(k_n + 2^{|M_n|})$, condition (4) is also satisfied.

The third proof is topological. Fix an enumeration $\{(n_i, w_i) : i \in \omega\}$ of all pairs (n, w) such that $n \in \omega$ and $w : n \rightarrow 2$. Define

$$R_i = \left\{ x \in (2^\omega)^{n_i} : \bigcap_{j \in n_i} x_j^{w_i(j)} \text{ is infinite} \right\}$$

for every $i \in \omega$ and observe that each R_i is comeager. By Exercise 8.8 and Theorem 19.1 in [13], there exists a comeager subset of the Vietoris hyperspace $K(2^\omega)$ consisting of perfect sets $P \subseteq 2^\omega$ such that $\{x \in P^{n_i} : x_j \neq x_k \text{ whenever } j \neq k\} \subseteq R_i$ for every $i \in \omega$. It is trivial to check that any such P is an independent family. \square

We remark that, in some sense, the last two proofs that we have given of the above lemma are the same. The Vietoris hyperspace $K(2^\omega)$ is naturally homeomorphic to the space X of pruned subtrees of ${}^{<\omega}2$ with basic open sets of the form $\{T \in X : T \cap {}^{<i}2 = \tau\}$ for a fixed pruned subtree τ of ${}^{<i}2$. Moreover, the set $\{T \in X : [T] \text{ is an independent family}\}$ is comeager in X because the combinatorial proof's rule for constructing T_{n+1} from T_n only needs to be followed infinitely often.

The authors propose to call the following *Kunen's closed embedding trick*.

Theorem 8 (Kunen). *Fix a zero-dimensional space C . There exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ that contains a homeomorphic copy of C as a closed subset.*

Proof. Fix P as in Lemma 7. Since P is homeomorphic to 2^ω , we can assume that C is a subspace of P . Observe that the family

$$\mathcal{G} = C \cup \{\omega \setminus x : x \in P \setminus C\}$$

has the finite intersection property because P is an independent family. Any non-principal ultrafilter $\mathcal{U} \supseteq \mathcal{G}$ will contain C as a closed subset. \square

Corollary 9. *There exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is not completely Baire.*

Proof. Simply choose $C = \mathbb{Q}$. \square

Since 2^ω is homeomorphic to $2^\omega \times 2^\omega$, one can easily obtain the following strengthening of Theorem 8. Observe that, since any space has at most \mathfrak{c} closed subsets, the result cannot be improved.

Theorem 10. *Fix a collection \mathcal{C} of zero-dimensional spaces such that $|\mathcal{C}| \leq \mathfrak{c}$. There exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ that contains a homeomorphic copy of C as a closed subset for every $C \in \mathcal{C}$.*

The next theorem, together with Corollary 9, shows that under $\text{MA}(\text{countable})$ the property of being completely Baire is enough to distinguish ultrafilters up to homeomorphism.

Theorem 11. *Assume that $\text{MA}(\text{countable})$ holds. Then there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is completely Baire.*

Proof. Enumerate as $\{Q_\eta : \eta \in \mathfrak{c}\}$ all subsets of 2^ω that are homeomorphic to \mathbb{Q} . By Lemma 6, it will be sufficient to construct a non-principal ultrafilter \mathcal{U} such that no Q_η is a closed subset of \mathcal{U} .

We will construct \mathcal{F}_ξ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, let \mathcal{U} be any ultrafilter extending $\bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. By induction, we will make sure that the following requirements are satisfied.

- (1) $\mathcal{F}_\mu \subseteq \mathcal{F}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
- (2) \mathcal{F}_ξ has the finite intersection property for every $\xi \in \mathfrak{c}$.
- (3) $|\mathcal{F}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
- (4) The potential closed copy of the rationals Q_η is dealt with at stage $\xi = \eta+1$: that is, either $\omega \setminus x \in \mathcal{F}_\xi$ for some $x \in Q_\eta$ or there exists $x \in \mathcal{F}_\xi$ such that $x \in \text{cl}(Q_\eta) \setminus Q_\eta$.

Start by letting $\mathcal{F}_0 = \text{Cof}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that \mathcal{F}_η is given. First assume that there exists $x \in Q_\eta$ such that $\mathcal{F}_\eta \cup \{\omega \setminus x\}$ has the finite intersection property. In this case, simply set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{\omega \setminus x\}$.

Now assume that $\mathcal{F}_\eta \cup \{\omega \setminus x\}$ does not have the finite intersection property for any $x \in Q_\eta$. It is easy to check that this implies $Q_\eta \subseteq \langle \mathcal{F}_\eta \rangle$. Apply Lemma 12 with $\mathcal{F} = \mathcal{F}_\eta$ and $Q = Q_\eta$ to get $x \in \text{cl}(Q_\eta) \setminus Q_\eta$ such that $\mathcal{F}_\eta \cup \{x\}$ has the finite intersection property. Finally, set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{x\}$. \square

Lemma 12. *Assume that $\text{MA}(\text{countable})$ holds. Let \mathcal{F} be a collection of subsets of ω with the finite intersection property such that $|\mathcal{F}| < \mathfrak{c}$. Let Q be a non-empty subset of 2^ω with no isolated points such that $Q \subseteq \langle \mathcal{F} \rangle$ and $|Q| < \mathfrak{c}$. Then there exists $x \in \text{cl}(Q) \setminus Q$ such that $\mathcal{F} \cup \{x\}$ has the finite intersection property.*

Proof. Consider the countable poset

$$\mathbb{P} = \{s \in {}^{<\omega}2 : \text{there exist } q \in Q \text{ and } n \in \omega \text{ such that } s = q \upharpoonright n\},$$

with the natural order given by reverse inclusion.

For every $\sigma = \{x_1, \dots, x_k\} \in [\mathcal{F}]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma, \ell} = \{s \in \mathbb{P} : \text{there exists } i \in \text{dom}(s) \setminus \ell \text{ such that } s(i) = x_1(i) = \dots = x_k(i) = 1\}.$$

Using the fact that $Q \subseteq \langle \mathcal{F} \rangle$, it is easy to see that each $D_{\sigma, \ell}$ is dense in \mathbb{P} .

For every $q \in Q$, define

$$D_q = \{s \in \mathbb{P} : \text{there exists } i \in \text{dom}(s) \text{ such that } s(i) \neq q(i)\}.$$

Since Q has no isolated points, each D_q is dense in \mathbb{P} .

Since $|\mathcal{F}| < \mathfrak{c}$ and $|Q| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_{\sigma, \ell} : \sigma \in [\mathcal{F}]^{<\omega}, \ell \in \omega\} \cup \{D_q : q \in Q\}$$

has also size less than \mathfrak{c} . Therefore, by $\text{MA}(\text{countable})$, there exists a \mathcal{D} -generic filter $G \subseteq \mathbb{P}$. Let $x = \bigcup G \in 2^\omega$. The dense sets of the form $D_{\sigma, \ell}$ ensure that $\mathcal{F} \cup \{x\}$ has the finite intersection property. The definition of \mathbb{P} guarantees that $x \in \text{cl}(Q)$. Finally, the dense sets of the form D_q guarantee that $x \notin Q$. \square

Question 1. Can the assumption that $\text{MA}(\text{countable})$ holds be dropped in Theorem 11?

4. COUNTABLE DENSE HOMOGENEITY

Definition 13. A space X is *countable dense homogeneous* if for every pair (D, E) of countable dense subsets of X there exists a homeomorphism $f : X \rightarrow X$ such that $f[D] = E$.

We will start this section by consistently constructing an ultrafilter that is not countable dense homogeneous. We will use Sierpiński's technique for killing homeomorphisms (see [19] or Appendix 2 of [5] for a nice introduction). The key lemma is the following.

Lemma 14. *Assume that $\text{MA}(\text{countable})$ holds. Let \mathcal{D} be a countable independent family that is dense in 2^ω . Fix D_1 and D_2 disjoint countable dense subsets of \mathcal{D} . Then there exists $\mathcal{A} \subseteq 2^\omega$ satisfying the following requirements.*

- \mathcal{A} is an independent family.
- $\mathcal{D} \subseteq \mathcal{A}$.
- If $G \supseteq \mathcal{D}$ is a G_δ subset of 2^ω and $f : G \rightarrow G$ is a homeomorphism such that $f[D_1] = D_2$, then there exists $x \in G$ such that $\{x, \omega \setminus f(x)\} \subseteq \mathcal{A}$.

Proof. Enumerate as $\{f_\eta : \eta \in \mathfrak{c}\}$ all homeomorphisms

$$f_\eta : G_\eta \rightarrow G_\eta$$

such that $f_\eta[D_1] = D_2$, where $G_\eta \supseteq \mathcal{D}$ is a G_δ subset of 2^ω .

We will construct \mathcal{A}_ξ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, set $\mathcal{A} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{A}_\xi$. By induction, we will make sure that the following requirements are satisfied.

- (1) $\mathcal{A}_\mu \subseteq \mathcal{A}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
- (2) \mathcal{A}_ξ is an independent family for every $\xi \in \mathfrak{c}$.
- (3) $|\mathcal{A}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
- (4) The homeomorphism f_η is dealt with at stage $\xi = \eta + 1$: that is, there exists $x \in G_\eta$ such that $\{x, \omega \setminus f_\eta(x)\} \subseteq \mathcal{A}_\xi$.

Start by letting $\mathcal{A}_0 = \mathcal{D}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that \mathcal{A}_η is given.

List as $\{w_\alpha : \alpha \in \kappa\}$ all the words in \mathcal{A}_η , where $\kappa = |\mathcal{A}_\eta| < \mathfrak{c}$ by (3). It is easy to check that, for any fixed $n \in \omega$, $\alpha \in \kappa$ and $\varepsilon_1, \varepsilon_2 \in 2$, the set

$$W_{\alpha, n, \varepsilon_1, \varepsilon_2} = \{x \in G_\eta : |w_\alpha \cap x^{\varepsilon_1} \cap f_\eta(x)^{\varepsilon_2}| \geq n\}$$

is open in G_η . It is also dense, because $D_1 \setminus (F \cup f_\eta^{-1}[F]) \subseteq W_{\alpha, n, \varepsilon_1, \varepsilon_2}$, where F consists of the finitely many elements of \mathcal{A}_η that appear in w_α . Therefore, each $W_{\alpha, n, \varepsilon_1, \varepsilon_2}$ is comeager in 2^ω . Recall that $\text{MA}(\text{countable})$ is equivalent to $\text{cov}(\mathcal{M}) = \mathfrak{c}$ (see Theorem 7.13 in [4] or Theorem 2.4.5 in [3]). It follows that the intersection

$$W = \bigcap \{W_{\alpha, n, \varepsilon_1, \varepsilon_2} : n \in \omega, \alpha \in \kappa \text{ and } \varepsilon_1, \varepsilon_2 \in 2\}$$

is non-empty. Now simply pick $x \in W$ and set $\mathcal{A}_\xi = \mathcal{A}_\eta \cup \{x, \omega \setminus f_\eta(x)\}$. \square

Theorem 15. *Assume that $\text{MA}(\text{countable})$ holds. Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is not countable dense homogeneous.*

Proof. Fix D_1, D_2 and \mathcal{A} as in Lemma 14. Let $\mathcal{U} \supseteq \mathcal{A}$ be any ultrafilter. Assume, in order to get a contradiction, that \mathcal{U} is countable dense homogeneous. Let $g : \mathcal{U} \rightarrow \mathcal{U}$ be a homeomorphism such that $g[D_1] = D_2$. By Lavrentiev's lemma, it is possible to extend g to a homeomorphism $f : G \rightarrow G$, where G is a G_δ subset of 2^ω (see Exercise 3.10 in [13]). By Lemma 14, there exists $x \in G$ such that $\{x, \omega \setminus f(x)\} \subseteq \mathcal{A} \subseteq \mathcal{U}$, contradicting the fact that $f(x) = g(x) \in \mathcal{U}$. \square

Question 2. Can the assumption that $\text{MA}(\text{countable})$ holds be dropped in Theorem 15?

When first trying to prove Theorem 15, we attempted to construct a non-principal ultrafilter \mathcal{U} such that no homeomorphism $g : \mathcal{U} \rightarrow \mathcal{U}$ would be such that $g[\text{Cof}] \cap \text{Cof} = \emptyset$. This is easily seen to be impossible by choosing g to be the multiplication by any coinfinite $x \in \mathcal{U}$. Actually, something much stronger holds by the following result of Van Mill (see Proposition 3.4 in [17]).

Definition 16 (Van Mill). A space X has the *separation property* if for every countable subset A of X and every meager subset B of X there exists a homeomorphism $f : X \rightarrow X$ such that $f[A] \cap B = \emptyset$.

Proposition 17 (Van Mill). *Let G be a Baire topological group acting on space X that is not meager in itself. Then, for all subsets A and B of X with A countable and B meager, the set of elements $g \in G$ such that $gA \cap B = \emptyset$ is dense in G .*

Corollary 18. *Every Baire topological group has the separation property.*

Corollary 19. *Every ultrafilter $\mathcal{U} \subseteq 2^\omega$ has the separation property.*

It is easy to see that, for Baire spaces, being countable dense homogeneous is stronger than having the separation property. On the other hand, the product of 2^ω and the one-dimensional sphere S^1 is a compact topological group that has the separation property but is not countable dense homogeneous (see Corollary 3.6 and Remark 3.7 in [17]). Theorem 15 consistently gives a zero-dimensional topological group with the same feature. Notice that such an example cannot be compact (or even Polish) by the following paragraph.

Recall that a space X is *strongly locally homogeneous* if it admits an open base \mathcal{B} such that whenever $U \in \mathcal{B}$ and $x, y \in U$ there exists a homeomorphism $f : X \rightarrow X$ such that $f(x) = y$ and $f \upharpoonright X \setminus U$ is the identity. For example, any homogeneous zero-dimensional space is strongly locally homogeneous. For Polish spaces, strong local homogeneity implies countable dense homogeneity (see Theorem 5.2 in [1]). In [18], Van Mill constructed a homogeneous Baire space that is strongly locally homogeneous but not countable dense homogeneous. Actually, his example does not even have the separation property (see Theorem 3.5 in [18]), so it cannot be a topological group by Corollary 18. In this sense, our example from Theorem 15 is better than his. On the other hand, his example is constructed in ZFC, while ours needs MA(countable). Furthermore, his example can be easily modified to have any given dimension (see Remark 4.1 in [18]).

Next, we will construct (still under MA(countable)) a non-principal ultrafilter that is countable dense homogeneous. In [2], Baldwin and Beaudoin used MA(countable) to construct a homogeneous Bernstein subset of 2^ω that is countable dense homogeneous. Both examples give a consistent answer to Question 389 in [9], which asks whether there exists a countable dense homogeneous space that is not completely metrizable. In [7], using metamathematical methods, Farah, Hrušák and Martínez Ranero showed that the answer to such question is ‘yes’ in ZFC.

The following lemma will be one of the key ingredients. The other key ingredient is the poset used in the proof of Lemma 22, which was inspired by the poset used in the proof of Lemma 3.1 in [2].

Lemma 20. *Let $f : 2^\omega \rightarrow 2^\omega$ be a homeomorphism. Fix a non-principal maximal ideal $\mathcal{J} \subseteq 2^\omega$ and a countable dense subset D of \mathcal{J} . Then f restricts to a homeomorphism of \mathcal{J} if and only if $\text{cl}(\{d + f(d) : d \in D\}) \subseteq \mathcal{J}$.*

Proof. Assume that f restricts to a homeomorphism of \mathcal{J} . It is easy to check that the function $g : 2^\omega \rightarrow 2^\omega$ defined by $g(x) = x + f(x)$ has range contained in \mathcal{J} . Since g is continuous, its range must be compact, hence closed in 2^ω .

Now assume that $\text{cl}(\{d + f(d) : d \in D\}) \subseteq \mathcal{J}$. Let $x \in 2^\omega$. Fix $d_n \in D$ for $n \in \omega$ so that $\lim_{n \rightarrow \infty} d_n = x$. By continuity,

$$x + f(x) = \lim_{n \rightarrow \infty} (d_n + f(d_n)) \in \mathcal{J}.$$

The proof is concluded by observing that if $a, b \in 2^\omega$ are such that $a + b \in \mathcal{J}$, then either $\{a, b\} \subseteq \mathcal{J}$ or $\{a, b\} \subseteq 2^\omega \setminus \mathcal{J}$. \square

Theorem 21. *Assume that MA(countable) holds. Then there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is countable dense homogeneous.*

Proof. For notational convenience, we will construct a maximal ideal $\mathcal{J} \subseteq 2^\omega$ containing all finite sets that is countable dense homogeneous. Enumerate as $\{(D_\eta, E_\eta) : \eta \in \mathfrak{c}\}$ all pairs of countable dense subsets of 2^ω .

We will construct \mathcal{I}_ξ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, let \mathcal{J} be any maximal ideal extending $\bigcup_{\xi \in \mathfrak{c}} \mathcal{I}_\xi$. By induction, we will make sure that the following requirements are satisfied.

- (1) $\mathcal{I}_\mu \subseteq \mathcal{I}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
- (2) \mathcal{I}_ξ has the finite union property for every $\xi \in \mathfrak{c}$.
- (3) $|\mathcal{I}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
- (4) The pair (D_η, E_η) is dealt with at stage $\xi = \eta + 1$: that is, either $\omega \setminus x \in \mathcal{I}_\xi$ for some $x \in D_\eta \cup E_\eta$ or there exists $x \in \mathcal{I}_\xi$ and a homeomorphism $f_\eta : 2^\omega \rightarrow 2^\omega$ such that $f_\eta[D_\eta] = E_\eta$ and $\{d + f_\eta(d) : d \in D_\eta\} \subseteq x \downarrow$.

Observe that, by Lemma 20, the second part of condition (4) guarantees that any maximal ideal \mathcal{J} extending \mathcal{I}_ξ will be such that $f_\eta : 2^\omega \rightarrow 2^\omega$ restricts to a homeomorphism of \mathcal{J} .

Start by letting $\mathcal{I}_0 = \text{Fin}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that \mathcal{I}_η is given. First assume that there exists $x \in D_\eta \cup E_\eta$ such that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ has the finite union property. In this case, we can just set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{\omega \setminus x\}$.

Now assume that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ does not have the finite union property for any $x \in D_\eta \cup E_\eta$. It is easy to check that this implies $D_\eta \cup E_\eta \subseteq \langle \mathcal{I}_\eta \rangle$. Let x and f be given by applying Lemma 22 with $\mathcal{I} = \mathcal{I}_\eta$, $D = D_\eta$ and $E = E_\eta$. Finally, set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{x\}$ and $f_\eta = f$. \square

Lemma 22. *Assume that MA(countable) holds. Let $\mathcal{I} \subseteq 2^\omega$ be a collection of subsets of ω with the finite union property and assume that $|\mathcal{I}| < \mathfrak{c}$. Fix two countable dense subsets D and E of 2^ω such that $D \cup E \subseteq \langle \mathcal{I} \rangle$. Then there exists a homeomorphism $f : 2^\omega \rightarrow 2^\omega$ and $x \in 2^\omega$ such that $f[D] = E$, $\mathcal{I} \cup \{x\}$ still has the finite union property and $\{d + f(d) : d \in D\} \subseteq x \downarrow$.*

Proof. Consider the countable poset \mathbb{P} consisting of all triples of the form $p = (s, g, \pi) = (s_p, g_p, \pi_p)$ such that, for some $n = n_p \in \omega$, the following requirements are satisfied.

- $s : n \rightarrow 2$.
- g is a bijection between a finite subset of D and a finite subset of E .
- π is a permutation of ${}^n 2$.

Furthermore, we require the following compatibility conditions to be satisfied. Condition (1) will actually ensure that $\{d + f(d) : d \in 2^\omega\} \subseteq x \downarrow$. Notice that this is equivalent to $(d + f(d))(i) \leq x(i)$ for all $d \in 2^\omega$ and $i \in \omega$.

- (1) $(t + \pi(t))(i) = 1$ implies $s(i) = 1$ for every $t \in {}^n 2$ and $i \in n$.
- (2) $\pi(d \upharpoonright n) = g(d) \upharpoonright n$ for every $d \in \text{dom}(g)$.

Order \mathbb{P} by declaring $q \leq p$ if the following conditions are satisfied.

- $s_q \supseteq s_p$.
- $g_q \supseteq g_p$.
- $\pi_q(t) \upharpoonright n_p = \pi_p(t \upharpoonright n_p)$ for all $t \in {}^{n_q} 2$.

For each $d \in D$, define

$$D_d^{\text{dom}} = \{p \in \mathbb{P} : d \in \text{dom}(g_p)\}.$$

Given $p \in \mathbb{P}$ and $d \in D \setminus \text{dom}(g_p)$, one can simply choose $e \in E \setminus \text{ran}(g_p)$ such that $e \upharpoonright n_p = \pi_p(d \upharpoonright n_p)$. This choice will make sure that $q = (s_p, g_p \cup \{(d, e)\}, \pi_p) \in \mathbb{P}$. Furthermore it is clear that $q \leq p$. So each D_d^{dom} is dense in \mathbb{P} .

For each $e \in E$, define

$$D_e^{\text{ran}} = \{p \in \mathbb{P} : e \in \text{ran}(g_p)\}.$$

As above, one can easily show that each D_e^{ran} is dense in \mathbb{P} .

For every $\sigma = \{x_1, \dots, x_k\} \in [\mathcal{I}]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma, \ell} = \{p \in \mathbb{P} : \text{there exists } i \in n_p \setminus \ell \text{ such that } s_p(i) = x_1(i) = \dots = x_k(i) = 0\}.$$

Next, we will prove that each $D_{\sigma, \ell}$ is dense in \mathbb{P} . So fix σ and ℓ as above. Let $p = (s, g, \pi) \in \mathbb{P}$ with $n_p = n$. Find $n' \geq \ell, n$ such that the following conditions hold.

- All $d \upharpoonright n'$ for $d \in \text{dom}(g)$ are distinct.
- All $e \upharpoonright n'$ for $e \in \text{ran}(g)$ are distinct.
- $x_1(n') = \dots = x_k(n') = d(n') = e(n') = 0$ for all $d \in \text{dom}(g)$, $e \in \text{ran}(g)$.

This is possible because \mathcal{I} has the finite union property and

$$\sigma \cup \text{dom}(g) \cup \text{ran}(g) \subseteq \langle \mathcal{I} \rangle.$$

We can choose a permutation π' of ${}^{n'} 2$ such that $\pi'(d \upharpoonright n') = g(d) \upharpoonright n'$ for every $d \in \text{dom}(g)$ and $\pi'(t) \upharpoonright n = \pi(t \upharpoonright n)$ for all $t \in {}^{n'} 2$. Extend s to $s' : n' \rightarrow 2$ by setting $s'(i) = 1$ for every $i \in [n, n')$. It is clear that $p' = (s', g, \pi') \in \mathbb{P}$ and $p' \leq p$.

Now let π'' be the permutation of ${}^{n'+1} 2$ obtained by setting

$$\pi''(t) = \pi'(t \upharpoonright n') \smallfrown t(n')$$

for all $t \in {}^{n'+1} 2$. Extend s' to $s'' : n' + 1 \rightarrow 2$ by setting $s''(n') = 0$. It is easy to check that $p'' = (s'', g, \pi'') \in D_{\sigma, \ell}$ and $p'' \leq p'$.

Since $|\mathcal{I}| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_{\sigma, \ell} : \sigma \in [\mathcal{I}]^{<\omega}, \ell \in \omega\} \cup \{D_d^{\text{dom}} : d \in D\} \cup \{D_e^{\text{ran}} : e \in E\}$$

has also size less than \mathfrak{c} . Therefore, by MA(countable), there exists a \mathcal{D} -generic filter $G \subseteq \mathbb{P}$. Define $x = \bigcup \{s_p : p \in G\}$. To define $f(y)(i)$, for a given $y \in 2^\omega$ and $i \in \omega$, choose any $p \in G$ such that $i \in n_p$ and set $f(y)(i) = \pi_p(y \upharpoonright n_p)(i)$. \square

Question 3. Can the assumption that MA(countable) holds be dropped in Theorem 21?

By Theorem 2.3 in [10], every analytic countable dense homogeneous space must be completely Baire. So the following question seems natural. See also Theorem 2.6 in [10].

Question 4. Is a countable dense homogeneous ultrafilter $\mathcal{U} \subseteq 2^\omega$ necessarily completely Baire?

5. A QUESTION OF HRUŠÁK AND ZAMORA AVILÉS

The main result of [10] states that, given a Borel subset X of 2^ω , the following statements are equivalent.

- X^ω is countable dense homogeneous.
- X is a G_δ .

Question 3.2 in the same paper asks whether there exists a non- G_δ subset X of 2^ω such that X^ω is countable dense homogeneous. By a rather straightforward modification of the proof of Theorem 21, we will give a consistent answer to such question (see Corollary 26).

Our example is also relevant to the second half of Question 387 in [9], which asks to characterize the zero-dimensional spaces X such that X^ω is countable dense homogeneous.

Observe that, given any ideal $\mathcal{I} \subseteq 2^\omega$, the infinite product \mathcal{I}^ω inherits the structure of topological group using coordinate-wise addition. The following lemma is proved exactly like the corresponding half of Lemma 20.

Lemma 23. *Let $f : (2^\omega)^\omega \rightarrow (2^\omega)^\omega$ be a homeomorphism. Fix a non-principal maximal ideal $\mathcal{J} \subseteq 2^\omega$ and a countable dense subset D of \mathcal{J}^ω . If $\text{cl}(\{d + f(d) : d \in D\}) \subseteq \mathcal{J}^\omega$ then f restricts to a homeomorphism of \mathcal{J}^ω .*

Theorem 24. *Assume that MA(countable) holds. Then there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that \mathcal{U}^ω is countable dense homogeneous.*

Proof. For notational convenience, we will construct a maximal ideal $\mathcal{J} \subseteq 2^\omega$ containing all finite sets such that \mathcal{J}^ω is countable dense homogeneous. Enumerate as $\{(D_\eta, E_\eta) : \eta \in \mathfrak{c}\}$ all pairs of countable dense subsets of $(2^\omega)^\omega$.

We will construct \mathcal{I}_ξ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, let \mathcal{J} be any maximal ideal extending $\bigcup_{\xi \in \mathfrak{c}} \mathcal{I}_\xi$. By induction, we will make sure that the following requirements are satisfied. Let $P_\eta = \bigcup_{i \in \omega} \pi_i[D_\eta \cup E_\eta]$, where $\pi_i : (2^\omega)^\omega \rightarrow 2^\omega$ is the natural projection.

- (1) $\mathcal{I}_\mu \subseteq \mathcal{I}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
- (2) \mathcal{I}_ξ has the finite union property for every $\xi \in \mathfrak{c}$.
- (3) $|\mathcal{I}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
- (4) The pair (D_η, E_η) is dealt with at stage $\xi = \eta + 1$: that is, either $\omega \setminus x \in \mathcal{I}_\xi$ for some $x \in P_\eta$ or there exists $x_i \in \mathcal{I}_\xi$ for every $i \in \omega$ and a homeomorphism $f_\eta : (2^\omega)^\omega \rightarrow (2^\omega)^\omega$ such that $f_\eta[D_\eta] = E_\eta$ and $\{d + f_\eta(d) : d \in D_\eta\} \subseteq \prod_{i \in \omega} (x_i \downarrow)$.

Observe that, by Lemma 23, the second part of condition (4) guarantees that any maximal ideal \mathcal{J} extending \mathcal{I}_ξ will be such that $f_\eta : (2^\omega)^\omega \rightarrow (2^\omega)^\omega$ restricts to a homeomorphism of \mathcal{J}^ω .

Start by letting $\mathcal{I}_0 = \text{Fin}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that \mathcal{I}_η is given. First assume that there exists $x \in P_\eta$ such

that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ has the finite union property. In this case, we can just set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{\omega \setminus x\}$.

Now assume that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ does not have the finite intersection property for any $x \in P_\eta$. It is easy to check that this implies $P_\eta \subseteq \langle \mathcal{I}_\eta \rangle$, hence $D_\eta \cup E_\eta \subseteq \langle \mathcal{I}_\eta \rangle^\omega$. Let x_i for $i \in \omega$ and f be given by applying Lemma 25 with $\mathcal{I} = \mathcal{I}_\eta$, $D = D_\eta$ and $E = E_\eta$. Finally, set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{x_i : i \in \omega\}$ and $f_\eta = f$. \square

Lemma 25. *Assume that MA(countable) holds. Let $\mathcal{I} \subseteq 2^\omega$ be a collection of subsets of ω with the finite union property and assume that $|\mathcal{I}| < \mathfrak{c}$. Fix two countable dense subsets D and E of $(2^\omega)^\omega$ such that $D \cup E \subseteq \langle \mathcal{I} \rangle^\omega$. Then there exists a homeomorphism $f : (2^\omega)^\omega \rightarrow (2^\omega)^\omega$ and $x_i \in 2^\omega$ for $i \in \omega$ such that $f[D] = E$, $\mathcal{I} \cup \{x_i : i \in \omega\}$ still has the finite union property and $\{d + f(d) : d \in D\} \subseteq \prod_{i \in \omega} (x_i \downarrow)$.*

Proof. We will make a natural identification of $(2^\omega)^\omega$ with $2^{\omega \times \omega}$. Namely, we will identify a sequence $(x_i)_{i \in \omega}$ with the function x given by $x(i, j) = x_i(j)$.

Consider the countable poset \mathbb{P} consisting of all triples of the form $p = (s, g, \pi) = (s_p, g_p, \pi_p)$ such that, for some $m = m_p \in \omega$ and $n = n_p \in \omega$, the following requirements are satisfied.

- $s : m \times n \rightarrow 2$.
- g is a bijection between a finite subset of D and a finite subset of E .
- π is a permutation of $m \times n$.

Furthermore, we require the following compatibility conditions to be satisfied. Condition (1) will actually ensure that $\{d + f(d) : d \in (2^\omega)^\omega\} \subseteq \prod_{i \in \omega} (x_i \downarrow)$. Notice that this is equivalent to $(d + f(d))(i, j) \leq x(i, j) = x_i(j)$ for all $d \in 2^{\omega \times \omega}$ and $(i, j) \in \omega \times \omega$.

- (1) $(t + \pi(t))(i, j) = 1$ implies $s(i, j) = 1$ for every $t \in m \times n$ and $(i, j) \in m \times n$.
- (2) $\pi(d \upharpoonright (m \times n)) = g(d) \upharpoonright (m \times n)$ for every $d \in \text{dom}(g)$.

Order \mathbb{P} by declaring $q \leq p$ if the following conditions are satisfied.

- $s_q \supseteq s_p$.
- $g_q \supseteq g_p$.
- $\pi_q(t) \upharpoonright (m_p \times n_p) = \pi_p(t \upharpoonright (m_p \times n_p))$ for all $t \in m_q \times n_q$.

For each $d \in D$, define

$$D_d^{\text{dom}} = \{p \in \mathbb{P} : d \in \text{dom}(g_p)\}.$$

Given $p \in \mathbb{P}$ and $d \in D \setminus \text{dom}(g_p)$, one can simply choose $e \in E \setminus \text{ran}(g_p)$ such that $e \upharpoonright (m_p \times n_p) = \pi_p(d \upharpoonright (m_p \times n_p))$. This choice will make sure that $q = (s_p, g_p \cup \{(d, e)\}, \pi_p) \in \mathbb{P}$. Furthermore it is clear that $q \leq p$. So each D_d^{dom} is dense in \mathbb{P} .

For each $e \in E$, define

$$D_e^{\text{ran}} = \{p \in \mathbb{P} : e \in \text{ran}(g_p)\}.$$

As above, one can easily show that each D_e^{ran} is dense in \mathbb{P} .

For each $\sigma = \{x_1, \dots, x_k\} \in [I]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma, \ell} = \{p \in \mathbb{P} : \text{there exists } j \in n_p \setminus \ell \text{ such that}$$

$$s_p(0, j) = \dots = s_p(m_p - 1, j) = x_1(j) = \dots = x_k(j) = 0\}.$$

Next, we will prove that each $D_{\sigma,\ell}$ is dense in \mathbb{P} . So fix σ and ℓ as above. Let $p = (s, g, \pi) \in \mathbb{P}$ with $m_p = m$ and $n_p = n$. Find $m' \geq m$ and $n' \geq \ell, n$ such that the following conditions hold.

- All $d \upharpoonright (m' \times n')$ for $d \in \text{dom}(g)$ are distinct.
- All $e \upharpoonright (m' \times n')$ for $e \in \text{ran}(g)$ are distinct.
- $x_1(n') = \dots = x_k(n') = d_i(n') = e_i(n') = 0$ for all $d \in \text{dom}(g)$, $e \in \text{ran}(g)$ and $i \in m'$.

This is possible because \mathcal{I} has the finite union property and

$$\sigma \cup \{d_i : d \in \text{dom}(g), i \in \omega\} \cup \{e_i : e \in \text{ran}(g), i \in \omega\} \subseteq \langle \mathcal{I} \rangle.$$

We can choose a permutation π' of $m' \times n'$ such that $\pi'(d \upharpoonright (m' \times n')) = g(d) \upharpoonright (m' \times n')$ for every $d \in \text{dom}(g)$ and $\pi'(t) \upharpoonright (m \times n) = \pi(t \upharpoonright (m \times n))$ for all $t \in m' \times n'$. Extend s to $s' : m' \times n' \rightarrow 2$ by setting $s'(i, j) = 1$ for every $(i, j) \in (m' \times n') \setminus (m \times n)$. It is clear that $p' = (s', g, \pi') \in \mathbb{P}$ and $p' \leq p$.

Now let π'' be the permutation of $m' \times (n'+1)$ obtained by setting

$$\pi''(t)(i, j) = \begin{cases} \pi'(t \upharpoonright (m' \times n'))(i, j) & \text{if } (i, j) \in m' \times n' \\ t(i, j) & \text{if } (i, j) \in m' \times \{n'\} \end{cases}$$

for all $t \in m' \times (n'+1)$. Extend s' to $s'' : m' \times (n'+1) \rightarrow 2$ by setting $s''(i, j) = 0$ for all $(i, j) \in m' \times \{n'\}$. It is easy to check that $p'' = (s'', g, \pi'') \in D_{\sigma,\ell}$ and $p'' \leq p'$.

We will need one last class of dense sets. For any given $\ell \in \omega$, define

$$D_\ell = \{p \in \mathbb{P} : m_p \geq \ell\}.$$

An easier version of the above argument shows that each D_ℓ is in fact dense.

Since $|\mathcal{I}| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_{\sigma,\ell} : \sigma \in [\mathcal{I}]^{<\omega}, \ell \in \omega\} \cup \{D_d^{\text{dom}} : d \in D\} \cup \{D_e^{\text{ran}} : e \in E\} \cup \{D_\ell : \ell \in \omega\}$$

has also size less than \mathfrak{c} . Therefore, by MA(countable), there exists a \mathcal{D} -generic filter $G \subseteq \mathbb{P}$. Define $x_i = \bigcup \{s_p(i, -) : p \in G\}$ for every $i \in \omega$. To define $f(y)(i, j)$, for a given $y \in 2^{\omega \times \omega}$ and $(i, j) \in \omega \times \omega$, choose any $p \in G$ such that $(i, j) \in m_p \times n_p$ and set $f(y)(i, j) = \pi_p(y \upharpoonright (m_p \times n_p))(i, j)$. \square

Corollary 26. *Assume that MA(countable) holds. Then there exists a non- G_δ subset X of 2^ω such that X^ω is countable dense homogeneous.*

Question 5. Can the assumption that MA(countable) holds be dropped in Theorem 24?

Question 6. Is there an analytic non- G_δ subset X of 2^ω such that X^ω is countable dense homogeneous? Co-analytic?

6. THE PERFECT SET PROPERTY

Definition 27. Let X be a space. We will say that $A \subseteq X$ has the *perfect set property* if A is either countable or it contains a perfect set.

It is a classical result of descriptive set theory, due to Souslin, that every analytic subset of a Polish space has the perfect set property (see, for example, Theorem 29.1 in [13]).

The following is an easy application of Kunen's closed embedding trick.

Theorem 28. *There exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ with a closed subset of cardinality \mathfrak{c} that does not have the perfect set property.*

Proof. Fix a Bernstein set B in 2^ω , then apply Theorem 8 with $C = B$. \square

Next, we will consistently construct a non-principal ultrafilter \mathcal{U} such that every closed subset of \mathcal{U} has the perfect set property. Actually, we will get a much stronger result (see Theorem 29).

Recall that a play of the *strong Choquet game* on a topological space (X, \mathcal{T}) is of the form

$$\begin{array}{ccccccc} \text{I} & (q_0, U_0) & & (q_1, U_1) & & \cdots & \\ \hline \text{II} & & V_0 & & V_1 & & \cdots, \end{array}$$

where $U_n, V_n \in \mathcal{T}$ are such that $q_n \in V_n \subseteq U_n$ and $U_{n+1} \subseteq V_n$ for every $n \in \omega$. Player II wins if $\bigcap_{n \in \omega} U_n \neq \emptyset$. The topological space (X, \mathcal{T}) is *strong Choquet* if II has a winning strategy in the above game. See Section 8.D in [13].

Define an *A-triple* to be a triple of the form (\mathcal{T}, A, Q) such that the following conditions are satisfied.

- \mathcal{T} is a strong Choquet, second-countable topology on 2^ω that is finer than the standard topology.
- $A \in \mathcal{T}$.
- Q is a non-empty countable subset of A with no isolated points in the subspace topology it inherits from \mathcal{T} .

By Theorem 25.18 in [13], for every analytic A there exists a topology \mathcal{T} as above. Also, by Exercise 25.19 in [13], such a topology \mathcal{T} necessarily consists only of analytic sets. In particular, all A-triples can be enumerated in type \mathfrak{c} .

Theorem 29. *Assume that $\text{MA}(\text{countable})$ holds. Then there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property for every analytic $A \subseteq 2^\omega$.*

Proof. Enumerate as $\{(\mathcal{T}_\eta, A_\eta, Q_\eta) : \eta \in \mathfrak{c}\}$ all A-triples, making sure that each triple appears cofinally often. Also, enumerate as $\{z_\eta : \eta \in \mathfrak{c}\}$ all subsets of ω .

We will construct \mathcal{F}_ξ for every $\xi \in \mathfrak{c}$ by transfinite recursion. By induction, we will make sure that the following requirements are satisfied.

- (1) $\mathcal{F}_\mu \subseteq \mathcal{F}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
- (2) \mathcal{F}_ξ has the finite intersection property for every $\xi \in \mathfrak{c}$.
- (3) $|\mathcal{F}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
- (4) By stage $\xi = \eta + 1$, we must have decided whether $z_\eta \in \mathcal{U}$: that is, $z_\eta^\varepsilon \in \mathcal{F}_\xi$ for some $\varepsilon \in 2$.
- (5) If $Q_\eta \subseteq \mathcal{F}_\eta$ then, at stage $\xi = \eta + 1$, we will deal with A_η : that is, there exists $x \in \mathcal{F}_\xi$ such that $x \upharpoonright \cap A_\eta$ contains a perfect subset.

In the end, let $\mathcal{U} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. Notice that \mathcal{U} will be an ultrafilter by (4).

Start by letting $\mathcal{F}_0 = \text{Cof}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that \mathcal{F}_η is given. First assume that $Q_\eta \not\subseteq \mathcal{F}_\eta$. In this case, simply set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{z_\eta^\varepsilon\}$ for a choice of $\varepsilon \in 2$ that is compatible with condition (2).

Now assume that $Q_\eta \subseteq \mathcal{F}_\eta$. Apply Lemma 30 with $\mathcal{F} = \mathcal{F}_\eta$, $A = A_\eta$, $Q = Q_\eta$ and $\mathcal{T} = \mathcal{T}_\eta$ to get a perfect set $P \subseteq A$ such that $\mathcal{F}_\eta \cup \{\bigcap P\}$ has the finite intersection property. Let $x = \bigcap P$. Set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{x, z_\eta^\varepsilon\}$, for some $\varepsilon \in 2$ compatible with condition (2).

Finally, we will check that \mathcal{U} has the required property. Assume that A is an analytic subset of 2^ω such that $A \cap \mathcal{U}$ is uncountable. By Theorem 25.18 in [13], there exists a second-countable, strong Choquet topology \mathcal{T} on 2^ω that is finer than the standard topology and contains A . Since every second countable, uncountable Hausdorff space contains a non-empty countable subspace with no isolated points, we can find such a subspace $Q \subseteq A \cap \mathcal{U}$. Since $\text{cf}(\mathfrak{c}) > \omega$, there exists $\mu \in \mathfrak{c}$ such that $Q \subseteq \mathcal{F}_\mu$. Since we listed each A-triple cofinally often, there exists $\eta \geq \mu$ such that $(\mathcal{T}, A, Q) = (\mathcal{T}_\eta, A_\eta, Q_\eta)$. Condition (5) guarantees that $\mathcal{U} \cap A$ will contain a perfect subset. \square

Lemma 30. *Assume that $\text{MA}(\text{countable})$ holds. Let \mathcal{F} be a collection of subsets of ω with the finite intersection property such that $|\mathcal{F}| < \mathfrak{c}$. Suppose that (\mathcal{T}, A, Q) is an A-triple with $Q \subseteq \mathcal{F}$. Then there exists a perfect subset P of A such that $\mathcal{F} \cup \{\bigcap P\}$ has the finite intersection property.*

Proof. Fix a winning strategy Σ for player II in the strong Choquet game in $(2^\omega, \mathcal{T})$. Also, fix a countable base \mathcal{B} for $(2^\omega, \mathcal{T})$. Let \mathbb{P} be the countable poset consisting of all functions p such that, for some $n = n_p \in \omega$, the following conditions hold.

- (1) $p : {}^{<n}2 \rightarrow Q \times \mathcal{B}$. We will use the notation $p(s) = (q_s^p, U_s^p)$.
- (2) $U_\emptyset^p = A$.
- (3) For every $s, t \in {}^{<n}2$, if s and t are incompatible (that is, $s \not\subseteq t$ and $t \not\subseteq s$) then $U_s^p \cap U_t^p = \emptyset$.
- (4) For every $s \in {}^{<n}2$,

$$\begin{array}{ccccccc} \text{I} & (q_{s \upharpoonright 0}^p, U_{s \upharpoonright 0}^p) & (q_{s \upharpoonright 1}^p, U_{s \upharpoonright 1}^p) & \cdots & (q_{s \upharpoonright n}^p, U_{s \upharpoonright n}^p) \\ \hline \text{II} & & V_{s \upharpoonright 0}^p & & V_{s \upharpoonright 1}^p & \cdots & V_{s \upharpoonright n}^p \end{array}$$

is a partial play of the strong Choquet game in $(2^\omega, \mathcal{T})$, where the open sets $V_{s \upharpoonright i}^p$ played by II are the ones dictated by the strategy Σ .

Order \mathbb{P} by setting $p \leq p'$ whenever $p \supseteq p'$.

For every $\ell \in \omega$, define

$$D_\ell = \{p \in \mathbb{P} : n_p \geq \ell\}.$$

Since Q has no isolated points and \mathcal{T} is Hausdorff, it is easy to see that each D_ℓ is dense.

For any fixed $\ell \in \omega$, consider the partition of 2^ω in clopen sets $\mathcal{P}_\ell = \{[s] : s \in {}^\ell 2\}$, then define

$$D_\ell^{\text{ref}} = \{p \in \mathbb{P} : \{U_s^p : s \in {}^{n_p} 2\} \text{ refines } \mathcal{P}_\ell\}.$$

Let us check that each D_ℓ^{ref} is dense. Given $p \in \mathbb{P}$ and $\ell \in \omega$, let $n = n_p$ and $q_s^0 = q_s^p$ for every $s \in {}^n 2$. Since Q has no isolated points, it is possible, for every $s \in {}^n 2$, to choose $q_s^1 \neq q_s^0$ such that $q_s^1 \in V_s^p \cap Q$. Then choose $j \geq \ell$ big enough so that $[q_s^1 \upharpoonright j] \cap [q_s^0 \upharpoonright j] = \emptyset$ for every $s \in {}^n 2$. Now simply extend p to a condition $p' : {}^{<n+1}2 \rightarrow Q \times \mathcal{B}$ by defining $p'(s \frown \varepsilon) = (q_s^\varepsilon, U_s^\varepsilon)$ for every $s \in {}^n 2$ and $\varepsilon \in 2$, where each $U_s^\varepsilon \in \mathcal{B}$ is such that $q_s^\varepsilon \in U_s^\varepsilon \subseteq V_s^p \cap [q_s^1 \upharpoonright j]$. It is easy to realize that $p' \in D_\ell^{\text{ref}}$.

For any fixed $\sigma = \{x_1, \dots, x_k\} \in [\mathcal{F}]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma, \ell} = \{p \in \mathbb{P} : \text{there exists } i \in \omega \setminus \ell \text{ such that}$$

$$x(i) = x_1(i) = \dots = x_k(i) = 1 \text{ for all } x \in U_s^p \text{ for all } s \in {}^{n_p} 2\}.$$

Let us check that each $D_{\sigma,\ell}$ is dense. Given $p \in \mathbb{P}$, σ and ℓ as above, let $n = n_p$ and $q_s^0 = q_s^p$ for every $s \in {}^n 2$. Notice that

$$\bigcap_{s \in {}^n 2} q_s^p \cap \bigcap \sigma$$

is an infinite subset of ω , because $Q \subseteq \mathcal{F}$ by assumption. So there exists $i \in \omega$ with $i \geq \ell$ such that

$$q_s^p(i) = x_1(i) = \cdots = x_k(i) = 1$$

for every $s \in {}^n 2$. Since Q has no isolated points, it is possible, for every $s \in {}^n 2$, to choose $q_s^1 \neq q_s^0$ such that $q_s^1 \in V_s^p \cap [q_s^0 \upharpoonright (i+1)] \cap Q$. Then choose $j \geq i+1$ big enough so that $[q_s^1 \upharpoonright j] \cap [q_s^0 \upharpoonright j] = \emptyset$ for every $s \in {}^n 2$. Now simply extend p to a condition $p' : {}^{<n+1} 2 \rightarrow Q \times \mathcal{B}$ by defining $p'(s \smallfrown \varepsilon) = (q_s^\varepsilon, U_s^\varepsilon)$ for every $s \in {}^n 2$ and $\varepsilon \in 2$, where each $U_s^\varepsilon \in \mathcal{B}$ is such that $q_s^\varepsilon \in U_s^\varepsilon \subseteq V_s^p \cap [q_s^\varepsilon \upharpoonright j]$. It is easy to realize that $p' \in D_{\sigma,\ell}$.

Since $|\mathcal{F}| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_\ell : \ell \in \omega\} \cup \{D_\ell^{\text{ref}} : \ell \in \omega\} \cup \{D_{\sigma,\ell} : \sigma \in [\mathcal{F}]^{<\omega}, \ell \in \omega\}$$

has also size less than \mathfrak{c} . Therefore, by MA(countable), there exists a \mathcal{D} -generic filter $G \subseteq \mathbb{P}$. Let $g = \bigcup G : {}^{<\omega} 2 \rightarrow Q \times \mathcal{B}$. Given $s \in {}^{<\omega} 2$, pick any $p \in G$ such that $s \in \text{dom}(p)$ and set $U_s = U_s^p$. For any $x \in 2^\omega$, since Σ is a winning strategy for Π , we must have $\bigcap_{n \in \omega} U_{x \upharpoonright n} \neq \emptyset$. Using the dense sets D_ℓ^{ref} , one can easily show that such intersection is actually a singleton. Therefore, letting $f(x)$ be the unique element of $\bigcap_{n \in \omega} U_{x \upharpoonright n}$ yields a well-defined function $f : 2^\omega \rightarrow A$. Using condition (3) in the definition of \mathbb{P} , one sees that f is injective.

Next, we will show that f is continuous in the standard topology, hence a homeomorphic embedding by compactness. Fix $x \in 2^\omega$ and let $y = f(x)$. Fix $\ell \in \omega$. Since G is a \mathcal{D} -generic filter, there must be $p \in D_\ell^{\text{ref}} \cap G$. Let $n = n_p$. Notice that this implies $U_{x \upharpoonright n} = U_{x \upharpoonright n}^p \subseteq [y \upharpoonright \ell]$, hence $f(x') \in [y \upharpoonright \ell]$ whenever $x' \in [x \upharpoonright n]$.

Therefore $P = \text{ran}(f)$ is a perfect subset of A . Finally, using the dense sets $D_{\sigma,\ell}$ one can show that $\mathcal{F} \cup \{\bigcap P\}$ has the finite intersection property. \square

Corollary 31. *Assume that MA(countable) holds. Then there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that every closed subset of \mathcal{U} has the perfect set property.*

Question 7. Can the assumption that MA(countable) holds be dropped in Theorem 29?

Observe that if $Q \subseteq 2^\omega$ is homeomorphic to \mathbb{Q} in the standard topology, $A = \text{cl}(Q)$ and \mathcal{T}_A is the topology obtained by declaring A open, then (\mathcal{T}_A, Q, A) is an A-triple because \mathcal{T}_A is Polish (see Lemma 13.2 in [13]). It follows easily that the ultrafilter constructed in Theorem 29 cannot contain closed copies of the rationals, hence it is completely Baire by Lemma 6.

Question 8. Is an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property whenever A is an analytic subset of 2^ω necessarily completely Baire?

We also remark that if $\Gamma \subseteq \mathcal{P}(2^\omega)$ is closed under c and \mathcal{U} is such that $A \cap \mathcal{U}$ has the perfect set property for all $A \in \Gamma$, then $A \setminus \mathcal{U}$ has the perfect set property for all $A \in \Gamma$.

7. EXTENDING THE PERFECT SET PROPERTY

Assuming $V = L$, there exists an uncountable co-analytic set A that does not contain any perfect set (see Theorem 25.37 in [12]). It follows that $\text{MA}(\text{countable})$ is not enough to extend Theorem 29 to all co-analytic sets. This section is devoted to attaining a positive result for the co-analytic case. Actually, we will obtain a much stronger result (see Theorem 35). We will need a modest large cardinal assumption, a larger fragment of MA , and the negation of CH .

Lemma 32. *Assume that $\mathcal{U} \subseteq 2^\omega$ is a P_{ω_2} -point. If $A \subseteq 2^\omega$ is such that every closed subspace of A has the perfect set property, then $A \cap \mathcal{U}$ has the perfect set property.*

Proof. Let A be as above, and assume that $A \cap \mathcal{U}$ is uncountable. Choose $B \subseteq A \cap \mathcal{U}$ such that $|B| = \omega_1$. Since \mathcal{U} is a P_{ω_2} -point, there is a pseudointersection x of B in \mathcal{U} . For some $n \in \omega$, uncountably many elements of B are in the closed set $C = (x \setminus n)^\uparrow$. By hypothesis, $A \cap C$ contains a perfect set P . We now have $A \cap \mathcal{U} \supseteq P$ as desired. Thus, $A \cap \mathcal{U}$ has the perfect set property. \square

It is not hard to verify that the hypothesis on A in the above lemma is optimal. Let x_0 and x_1 be complementary infinite subsets of ω . Identify each $\mathcal{P}(x_i)$ with the perfect set $\{x \in 2^\omega : x(n) = 0 \text{ for all } n \in x_{1-i}\}$. Fix a Bernstein subset B_i of $\mathcal{P}(x_i)$ and set $A_i = B_i \cup \mathcal{P}(x_{1-i})$ for each $i \in 2$. Each A_i has the perfect set property. However, if $\mathcal{U} \subseteq 2^\omega$ is an ultrafilter, then some $A_i \cap \mathcal{U}$ lacks the perfect set property. Indeed, if $x_i \in \mathcal{U}$, then $y \in \mathcal{U}$ for some subset $y \subseteq x_i$ such that $x_i \setminus y$ is infinite. The perfect set $y^\uparrow \cap \mathcal{P}(x_i)$ contains \mathfrak{c} many elements of B_i , so $A_i \cap \mathcal{U}$ has size \mathfrak{c} as well. However, $A_i \cap \mathcal{U} \subseteq B_i$, so $A_i \cap \mathcal{U}$ does not contain a perfect set.

The following lemma is essentially due to Ihoda (Judah) and Shelah (see Theorem 3.1 in [11]). Given a class Γ , we define $\text{PSP}(\Gamma)$ to mean that every $X \in \Gamma \cap \mathcal{P}(2^\omega)$ has the perfect set property.

Lemma 33. *The existence of a Mahlo cardinal is equiconsistent with*

$$\text{MA}(\sigma\text{-centered}) + \neg\text{CH} + \text{PSP}(L(\mathbb{R})).$$

Proof. Any generic extension by the Levy collapse $\text{Col}(\omega, \kappa)$ of an inaccessible cardinal κ to ω_1 satisfies $\text{PSP}(L(\mathbb{R}))$ (see the proof of Theorem 11.1 in [14]). By the proof of Lemma 1.1 in [11], if κ is inaccessible and \mathbb{P} is a forcing poset that satisfies the following conditions, then every generic extension $V[G]$ of V by \mathbb{P} is such that $L(\mathbb{R})^{V[G]} = L(\mathbb{R})^{V[H]}$ for some V -generic filter $H \subseteq \text{Col}(\omega, \kappa)$.

- (1) \mathbb{P} has the κ -cc.
- (2) \mathbb{P} forces $\kappa = \omega_1$.
- (3) For every $R \subseteq \mathbb{P}$ of size less than κ , there exists $Q \subseteq \mathbb{P}$ such that $|Q| < \kappa$, $R \subseteq Q$, and Q is completely embedded in \mathbb{P} by the inclusion map.

Assuming that there exists a Mahlo cardinal κ , the proof of Theorem 3.1 in [11] constructs a generic extension $V[G]$ of V by a forcing \mathbb{P} such that $V[G]$ satisfies $\text{MA}(\sigma\text{-centered}) + \neg\text{CH}$, using a forcing poset \mathbb{P} that satisfies conditions (1), (2) and (3). Therefore, $\text{PSP}(L(\mathbb{R}))$ also holds in $V[G]$.

Conversely, $\text{PSP}(L(\mathbb{R}))$ implies that all injections of ω_1 into 2^ω are outside of $L(\mathbb{R})$, which in turn implies $\omega_1^{L[r]} < \omega_1$ for all reals r . The proof of Theorem 3.1 in [11] shows that if $\text{MA}(\sigma\text{-centered}) + \neg\text{CH}$ holds and $\omega_1^{L[r]} < \omega_1$ for all reals r , then ω_1 is Mahlo in L . \square

For the convenience of the reader, we include the proof of the following standard lemma.

Lemma 34. *Assume that $\text{MA}(\sigma\text{-centered})$ holds. Then there exists a $\text{P}_\mathfrak{c}$ -point \mathcal{U} .*

Proof. Enumerate all subsets of ω as $\{z_\eta : \eta \in \mathfrak{c}\}$. We will construct \mathcal{F}_ξ for every $\xi \in \mathfrak{c}$ by transfinite recursion. By induction, we will make sure that the following requirements are satisfied.

- (1) $\mathcal{F}_\mu \subseteq \mathcal{F}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
- (2) \mathcal{F}_ξ has the finite intersection property for every $\xi \in \mathfrak{c}$.
- (3) $|\mathcal{F}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
- (4) By stage $\xi = \eta + 1$, we must have decided whether $z_\eta \in \mathcal{U}$: that is, $z_\eta^\varepsilon \in \mathcal{F}_\xi$ for some $\varepsilon \in 2$.
- (5) At stage $\xi = \eta + 1$, we will make sure that \mathcal{F}_ξ contains a pseudointersection of \mathcal{F}_η .

Start by letting $\mathcal{F}_0 = \text{Cof}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that \mathcal{F}_η is given.

Since $\text{MA}(\sigma\text{-centered})$ implies $\mathfrak{p} = \mathfrak{c}$ (see Theorem 7.12 in [4]), there exists an infinite pseudointersection x of \mathcal{F}_η . Now simply set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{x, z_\eta^\varepsilon\}$ for a choice of $\varepsilon \in 2$ that is compatible with condition (2).

In the end, let $\mathcal{U} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. Notice that \mathcal{U} will be an ultrafilter by (4). Since $\mathfrak{p} = \mathfrak{c}$ is regular (see Theorem 7.15 in [4]), condition (5) implies that \mathcal{U} is a $\text{P}_\mathfrak{c}$ -point. \square

It is well-known that $\text{MA}(\text{countable})$ is not a sufficient hypothesis for the above lemma. Consider the Cohen model $W = V[(c_\alpha : \alpha < \omega_2)]$, where $V \models \text{CH}$ and each c_α is an element of 2^ω that avoids all meager Borel sets with Borel codes in $V[(c_\beta : \beta < \omega_2, \beta \neq \alpha)]$. Observe that every $x \in 2^\omega$ is in $V[(c_\alpha : \alpha \in I)]$ for some countable set $I \subseteq \omega_2$. In this model, $\text{cov}(\mathcal{M}) = \mathfrak{c} = \omega_2$, so $\text{MA}(\text{countable}) + \neg\text{CH}$ holds (see Theorem 7.13 in [4]). However, if $\mathcal{U} \in W$ is a non-principal ultrafilter, then $\mathcal{U} \cap V[(c_\alpha : \alpha < \omega_1)]$ is a subset of \mathcal{U} of size ω_1 with no infinite pseudointersection.

Theorem 35. *It is consistent, relative to a Mahlo cardinal, that there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property for all $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$. On the other hand, if there exists such an ultrafilter \mathcal{U} , then ω_1 is inaccessible in L .*

Proof. Assume that $\text{MA}(\sigma\text{-centered}) + \neg\text{CH} + \text{PSP}(L(\mathbb{R}))$ holds, which is consistent relative to a Mahlo cardinal by Lemma 33. By Lemma 34, there exists a $\text{P}_\mathfrak{c}$ -point \mathcal{U} . Since $\neg\text{CH}$ holds, \mathcal{U} is a P_{ω_2} -point. Fix $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$. Every closed subspace C of A is also in $L(\mathbb{R})$ because $C = A \cap [T]$ for some tree $T \subseteq 2^{<\omega}$. By $\text{PSP}(L(\mathbb{R}))$, all such C have the perfect set property. So $A \cap \mathcal{U}$ has the perfect set property by Lemma 32.

For the second half of the theorem, assume that $\mathcal{U} \subseteq 2^\omega$ is a non-principal ultrafilter such that $A \cap \mathcal{U}$ has the perfect set property for all $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$. First, observe that given A as above, $c[A]$ is in $L(\mathbb{R})$ too, so $A \cap \mathcal{U}$ and $c[A] \cap \mathcal{U}$ have the perfect set property. Since

$$A = (A \cap \mathcal{U}) \cup (A \cap c[\mathcal{U}]) = (A \cap \mathcal{U}) \cup c[c[A] \cap \mathcal{U}],$$

it follows that A itself has the perfect set property. So $\text{PSP}(L(\mathbb{R}))$ holds, which implies $\omega_1^{L[r]} < \omega_1$ for all reals r . Therefore ω_1 is inaccessible in L . \square

Question 9. What is the exact consistency strenght of a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property for all $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$? In particular, does the Levy collapse $\text{Col}(\omega, \kappa)$ of an inaccessible cardinal κ to ω_1 force such an ultrafilter?

8. P-POINTS

Given a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$, it seems natural to investigate whether there is any relation between the topological properties of \mathcal{U} that we studied so far and combinatorial properties of \mathcal{U} . In order to construct several kinds of non-P-points, we will essentially use an idea from [15].

Definition 36. A *mixed independent family* is a pair $(\mathcal{F}, \mathcal{A})$ of collections of subsets of ω such that

$$\bigcap \sigma \cap \bigcap_{x \in \tau} x^{w(x)}$$

is infinite whenever $\sigma \in [\mathcal{F}]^{<\omega}$, $\tau \in [\mathcal{A}]^{<\omega}$ and $w : \tau \rightarrow 2$. A *dual mixed independent family* is a pair $(\mathcal{I}, \mathcal{B})$ of collections of subsets of ω such that $(c[\mathcal{I}], c[\mathcal{B}])$ is a mixed independent family.

Lemma 37. Let $(\mathcal{F}, \mathcal{A})$ be a mixed independent family such that \mathcal{A} is infinite. Then there exists a non-P-point \mathcal{U} extending $\mathcal{F} \cup \mathcal{A}$.

Proof. Fix a countably infinite subset \mathcal{B} of \mathcal{A} . It is easy to check that

$$\mathcal{G} = \mathcal{F} \cup \mathcal{A} \cup \{\omega \setminus x : x \subseteq^* y \text{ for every } y \in \mathcal{B}\}$$

has the finite intersection property. Let \mathcal{U} be any ultrafilter extending \mathcal{G} . It is clear that \mathcal{B} has no pseudointersection in \mathcal{U} . \square

Similarly, one can prove the following.

Lemma 38. Let $(\mathcal{I}, \mathcal{B})$ be a dual mixed independent family such that \mathcal{B} is infinite. Then there exists a maximal ideal \mathcal{J} extending $\mathcal{I} \cup \mathcal{B}$ that is not a P-ideal.

We will begin by studying the relation between P-points and completely Baire ultrafilters.

Theorem 39. There exists a non-P-point $\mathcal{U} \subseteq 2^\omega$ that is not completely Baire.

Proof. We will use the same notation as in the proof of Theorem 8. Choose $C = \mathbb{Q}$, so that any ultrafilter extending \mathcal{G} will contain a closed copy of \mathbb{Q} . Now simply apply Lemma 37 to (\emptyset, \mathcal{G}) . \square

Theorem 40. Assume that $\text{MA}(\text{countable})$ holds. Then there exists a P-point $\mathcal{U} \subseteq 2^\omega$ that is completely Baire.

Proof. Enumerate all countable collections of subsets of ω as $\{\mathcal{C}_\eta : \eta \in \mathfrak{c}\}$. The setup of the construction will be as in the proof of Theorem 11, but we will do different things at even and odd successor stages.

Start by letting $\mathcal{F}_0 = \text{Cof}$. Take unions at limit stages. At a successor stage $\xi = 2\eta + 1$, assume that $\mathcal{F}_{2\eta}$ is given, then take care of \mathcal{C}_η as in the proof of Theorem 11. At a successor stage $\xi = 2\eta + 2$, assume that $\mathcal{F}_{2\eta+1}$ is given, then take care of \mathcal{C}_η as follows.

First assume that there exists $x \in \mathcal{C}_\eta$ such that $\mathcal{F}_{2\eta+1} \cup \{\omega \setminus x\}$ has the finite intersection property. In this case, we can just set $\mathcal{F}_\xi = \mathcal{F}_{2\eta+1} \cup \{\omega \setminus x\}$. Now assume

that $\mathcal{F}_{2\eta+1} \cup \{\omega \setminus x\}$ does not have the finite intersection property for any $x \in \mathcal{C}_\eta$. It is easy to check that this implies $\mathcal{C}_\eta \subseteq \langle \mathcal{F}_{2\eta+1} \rangle$. Recall that $\text{MA}(\text{countable})$ implies $\mathfrak{d} = \mathfrak{c}$ (see, for example, Proposition 5.5 and Theorem 7.13 in [4]). So, by Proposition 6.24 in [4], there exists a pseudointersection x of \mathcal{C}_η such that $\mathcal{F}_{2\eta+1} \cup \{x\}$ has the finite intersection property. Finally, set $\mathcal{F}_\xi = \mathcal{F}_{2\eta+1} \cup \{x\}$. \square

Question 10. For a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$, is being a P-point equivalent to being completely Baire?

Now we turn to the relation between P-points and countable dense homogeneous ultrafilters.

Theorem 41. *Assume that $\text{MA}(\text{countable})$ holds. Then there exists a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is countable dense homogeneous but not a P-point.*

Proof. For notational convenience, we will actually construct a maximal ideal $\mathcal{J} \subseteq 2^\omega$ that is countable dense homogeneous but not a P-ideal.

The setup of the construction will be as in the proof of Theorem 21, but we will simultaneously construct \mathcal{B}_ξ for $\xi \in \mathfrak{c}$ so that the following conditions will be satisfied. In the end, set $\mathcal{B} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{B}_\xi$ and apply Lemma 38 to $(\mathcal{I}, \mathcal{B})$.

- (1) $\mathcal{B}_\mu \subsetneq \mathcal{B}_\eta$ whenever $\mu < \eta < \mathfrak{c}$.
- (2) $(\mathcal{I}_\xi, \mathcal{B}_\xi)$ is a dual mixed independent family for every $\xi \in \mathfrak{c}$.
- (3) $|\mathcal{B}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.

Start by letting $(\mathcal{I}_0, \mathcal{B}_0) = (\text{Fin}, \emptyset)$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that $(\mathcal{I}_\eta, \mathcal{B}_\eta)$ is given. First get x by applying Lemma 42 with $\mathcal{I} = \mathcal{I}_\eta$, $\mathcal{B} = \mathcal{B}_\eta$ and $(D, E) = (D_\eta, E_\eta)$. Then, as in the proof of Lemma 14, use $\text{MA}(\text{countable})$ to get $y \notin \mathcal{B}_\eta$ such that $(\mathcal{I}_\eta \cup \{x\}, \mathcal{B}_\eta \cup \{y\})$ is still a dual mixed independent family. Finally, set $(\mathcal{I}_\xi, \mathcal{B}_\xi) = (\mathcal{I}_\eta \cup \{x\}, \mathcal{B}_\eta \cup \{y\})$. \square

The following lemma is easily proved by modifying the proof of Lemma 22 (substitute the dense sets $D_{\sigma, \ell}$ with the obviously defined dense sets $D_{\sigma, \tau, w, \ell}$).

Lemma 42. *Assume that $\text{MA}(\text{countable})$ holds. Let $(\mathcal{I}, \mathcal{B})$ be a dual mixed independent family such that $|\mathcal{I}| < \mathfrak{c}$ and $|\mathcal{B}| < \mathfrak{c}$. Fix two countable dense subsets D and E of 2^ω such that $D \cup E \subseteq \langle \mathcal{I} \rangle$. Then there exists a homeomorphism $f : 2^\omega \rightarrow 2^\omega$ and $x \in 2^\omega$ such that $f[D] = E$, $(\mathcal{I} \cup \{x\}, \mathcal{B})$ is still a dual mixed independent family and $\{d + f(d) : d \in D\} \subseteq x \downarrow$.*

Theorem 43. *Assume that $\text{MA}(\text{countable})$ holds. Then there exists a non-P-point $\mathcal{U} \subseteq 2^\omega$ that is not countable dense homogeneous.*

Proof. Let \mathcal{A} be as in Lemma 14. By the proof of Theorem 15, no ultrafilter extending \mathcal{A} is countable dense homogeneous. Now simply apply Lemma 37 to (\emptyset, \mathcal{A}) . \square

Theorem 44. *Assume that $\text{MA}(\text{countable})$ holds. Then there exists a P-point $\mathcal{U} \subseteq 2^\omega$ that is countable dense homogeneous.*

Proof. For notational convenience, we will actually construct a maximal ideal $\mathcal{J} \subseteq 2^\omega$ that is countable dense homogeneous and a P-ideal. Enumerate all countable collections of subsets of ω as $\{\mathcal{C}_\eta : \eta \in \mathfrak{c}\}$. The setup of the construction will be as in the proof of Theorem 21, but we will do different things at even and odd successor stages.

Start by letting $\mathcal{I}_0 = \text{Fin}$. Take unions at limit stages. At a successor stage $\xi = 2\eta + 1$, assume that $\mathcal{I}_{2\eta}$ is given, then take care of (D_η, E_η) as in the proof of Theorem 21. At a successor stage $\xi = 2\eta + 2$, assume that $\mathcal{I}_{2\eta+1}$ is given, then take care of \mathcal{C}_η as follows.

First assume that there exists $x \in \mathcal{C}_\eta$ such that $\mathcal{I}_{2\eta+1} \cup \{\omega \setminus x\}$ has the finite union property. In this case, we can just set $\mathcal{I}_\xi = \mathcal{I}_{2\eta+1} \cup \{\omega \setminus x\}$. Now assume that $\mathcal{I}_{2\eta+1} \cup \{\omega \setminus x\}$ does not have the finite union property for any $x \in \mathcal{C}_\eta$. It is easy to check that this implies $\mathcal{C}_\eta \subseteq \langle \mathcal{I}_{2\eta+1} \rangle$. As in the proof of Theorem 40, it is possible to get a pseudounion x of \mathcal{C}_η such that $\mathcal{I}_{2\eta+1} \cup \{x\}$ has the finite union property. Finally, set $\mathcal{I}_\xi = \mathcal{I}_{2\eta+1} \cup \{x\}$. \square

Question 11. Is a P-point $\mathcal{U} \subseteq 2^\omega$ necessarily countable dense homogeneous?

Finally, we will investigate the relation between P-points and the perfect set property.

Theorem 45. *There exists a non-P-point $\mathcal{U} \subseteq 2^\omega$ with a closed subset of cardinality \mathfrak{c} that does not have the perfect set property.*

Proof. We will use the same notation as in the proof of Theorem 8. Choose C to be a Bernstein set in 2^ω , so that any ultrafilter extending \mathcal{G} will have a closed subset without the perfect property. Now simply apply Lemma 37 to (\emptyset, \mathcal{G}) . \square

Theorem 46. *Assume that $\text{MA}(\text{countable})$ holds. Then there exists a P-point $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property whenever A is an analytic subset of 2^ω .*

Proof. Enumerate all countable collections of subsets of ω as $\{\mathcal{C}_\eta : \eta \in \mathfrak{c}\}$. The setup of the construction will be as in the proof of Theorem 29, but we will do different things at even and odd successor stages.

Start by letting $\mathcal{F}_0 = \text{Cof}$. Take unions at limit stages. At a successor stage $\xi = 2\eta + 1$, assume that $\mathcal{F}_{2\eta}$ is given, then take care of $(\mathcal{T}_\eta, A_\eta, Q_\eta)$ and z_η as in the proof of Theorem 29. At a successor stage $\xi = 2\eta + 2$, assume that $\mathcal{F}_{2\eta+1}$ is given, then take care of \mathcal{C}_η as in the proof of Theorem 40. \square

Question 12. For a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$, is being a P-point equivalent to $A \cap \mathcal{U}$ having the perfect set property whenever A is an analytic subset of 2^ω ?

Observe that Lemma 32 might be viewed as a partial answer to Question 12.

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